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Report 1079

DEPARTMENT OF THE NAVY

# DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

o

AERODYNAMICS

o

STRUCTURAL MECHANICS

o

APPLIED MATHEMATICS

## TRIPPING OF T-SHAPED STIFFENING RINGS ON CYLINDERS UNDER EXTERNAL PRESSURE

by

E.H. Kennard



RESEARCH AND DEVELOPMENT REPORT

November 1959

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## TABLE OF CONTENTS



## LIST OF FIGURES

# NOTATION

| | |
|---|---|
| $A_f$ | Cross-sectional area of flange |
| $A_w$ | Cross-sectional area of web |
| $c$ | Depth of web |
| $D$ | Web stiffness, defined in Equation [8] |
| $e$ | See Equations [14a, d] |
| $E$ | Young's modulus |
| $F$ | Flange stiffness parameter, Equation [31a] |
| $g$ | See Equation [14c] |
| $h$ | Shell thickness |
| $I_f$ | Areal moment of inertia of flange cross section, Equation [27a] |
| $J_f$ | Flange torsional rigidity (torsional moment divided by rate of twist) divided by $E$ |
| $k_1$, $k_2$, etc. | Constants defined by Formulas [24] |
| $K_1$ | Circumferential thrust in web divided by $c$ |
| $K_f$ | Circumferential thrust in flange |
| $L$ | Numerical value of radial force per unit length in web |
| $L_0$ | Value of $L$ at toe of web ($z = 0$) |
| $M_b$ | Bending moment in flange |
| $M_s$ | Bending moment in web on section perpendicular to $s$, per radial unit length |
| $M_t$ | Torsional moment in flange |
| $M_z$ | Bending moment in web on section perpendicular to $z$, per circumferential unit length |
| $M_{zs}$ | Twisting moment in web per unit length |
| $M'$ | Value of $M_z$ at $z = c$ |
| $M_0$ | Value of $M_z$ at $z = 0$ |
| $M_s'$ | Value of $M_s$ at $z = c$ |
| $n$ | Number of waves around circumference, variable quantities varying as cos $(ns/R)$ or sin $(ns/R)$ |

| | |
|---|---|
| $p$ | Pressure on shell |
| $Q$ | Axial force on flange cross section |
| $Q_s$ | Axial force on web cross section perpendicular to $s$, per radial unit length |
| $Q_z$ | Axial force on web cross section perpendicular to $z$, per circumferential unit length |
| $Q'$ | Value of $Q_z$ at $z = c$ |
| $R$ | Radius of cylinder at toe of web |
| $R_f$ | Radius of flange (without systematic differentiation between centroidal circle and circle of attachment to web) |
| $s$ | Circumferential coordinate in web, which is treated approximately as a straight strip |
| $t_f$ | Thickness of flange |
| $t_w$ | Thickness of web |
| $u(z, s)$ | Elastic displacement of web (in axial direction) |
| $u_0(z, s)$ | Initial axial displacement of web |
| $u'(s)$ | Value of $u$ at $z = c$ |
| $u_0'(s)$ | Value of $u_0$ at $z = c$ |
| $W$ | $= D/c^2$, web stiffness parameter |
| $y$ | $= z/c$ |
| $z$ | Radial coordinate in web, measured away from surface of cylinder ($z = 0$) |
| $\alpha$ | Defined so that radial load per unit length on flange is $(1 - \alpha)L_0$ |
| $\beta$ | $= c/R$ |
| $\beta_f$ | $= c/R_f$ |
| $\epsilon$ | See Equations [14b, e] |
| $\theta(s)$ | Assumed initial slope of web |
| $\mu$ | $= \beta_f W/F$ |
| $\nu$ | Poisson's ratio |
| $\sigma$ | Stress, defined as needed (as in Equations [35] and [36a,b]) |
| $\tau$ | $= J_f/I_f$ |
| $\omega$ | $= \partial u/\partial z$ at $z = c$ |

## ABSTRACT

An approximate theory is developed for estimating the additional stresses caused by an initial tilt of a T-shaped stiffening ring located either inside or outside of a cylinder under uniform pressure, and the buckling load for an ideal ring, without restriction to axisymmetric symmetry. In practical cases the results appear not to differ markedly from those of the axisymmetric case, which was treated ~~in David Taylor Model Basin Report 1078.~~ previously, (AD-115 472).


## INTRODUCTION

Although, in designing the stiffening rings for submarine hulls, major attention is usually given to the strength of the ring in its own plane, there is also a possibility that a ring may *trip*, that is, it may buckle or deform laterally. If such deformations become large, the support furnished by the ring to the cylindrical hull may be seriously impaired. In present design, precautions against tripping are taken, in effect, by treating the rings as if they were straight stiffeners on a panel loaded in compression and there is little evidence that this method results in designs that are inadequate. Nevertheless, a specific analysis of the effect of ring curvature was considered worthwhile.

The case of axisymmetric tripping, which is vastly simpler than the general case, was treated in a separate report.[1] Axisymmetric tripping, however, can occur only in the case of an inside ring. The present report records an approximate treatment of tripping with the formation of circumferential waves. The results obtained suggest that inside stiffening rings as now designed are probably at least not much weaker toward lateral deflection in waves than they are toward axisymmetric lateral deformation.

In the case of beams, the ideal case of the buckling of a perfectly uniform and straight beam is of great practical interest. The critical load for lateral buckling of a stiffening ring of current proportions, on the other hand, is so high as to have little practical significance. Nevertheless, a certain interest may attach to the ideal case of the buckling of a perfect ring because of the light thus thrown on the elastic phenomena in stiffening rings.

In the axisymmetric case, buckling results from column buckling of the web caused by radial compressive force, which can occur only in an inside ring. When the buckling deformation is not symmetrical around the ring axis, the circumferential thrust also promotes buckling, whether the ring is inside or outside of the cylinder. If the flange were absent, the web alone would buckle in squarish waves under a radial load much smaller that that required for axisymmetric buckling. The flange by itself, on the other hand, is relatively weak and favors few waves. The stiffener flange on an SS 212 submarine, for example, taken by itself as a flat hoop, would buckle in four waves ($n = 2$) under a radial load of less than 6 lb/in.

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[1] References are listed on page 33.

In the combined system of web and flange there is thus a competition between the preference of the web for many waves and that of the flange for few. There occurs also an important incompatibility of the buckling patterns as favored by the web and the flange, and this tends to make the buckling load higher than it would be for either web or flange alone.

More important is the estimation of the stresses evoked in a stiffening ring that is imperfectly shaped; this was the principal topic in Reference 1. The treatment is extended in the present report to a simple tilt that varies around the ring.

The circle of attachment of the web of the ring to the shell will be assumed to remain circular at all times, since analysis indicates that small deformations of this circle in its own plane, although important in their own right, would have no first-order effect upon the phenomenon of tripping. Thus tripping of the ring can be treated independently without entering upon the general subject of the buckling failure of stiffened cylinders.

## THE LOAD ON THE RING

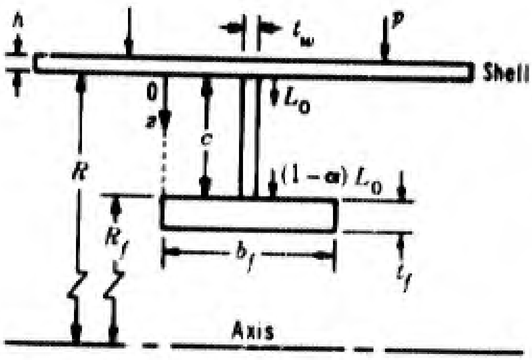

Figure 1 – Dimensions of Internal T-Stiffener

The problem to be considered is that of a stiffening ring on a long cylindrical shell subjected to external pressure. In developing the analysis, the ring will be assumed to be located on the *inside* of the cylinder, since that is the most unstable location; the simple changes required to adapt the formulas to outside rings will be indicated at appropriate points. Furthermore, the ring will be assumed to be a simple T (Figure 1); if it has in reality a faying flange, this will be treated for present purposes as part of the cylindrical shell.

The uniform external pressure $p$ applied to the shell, by way of elastic reactions evoked in the cylinder, loads the ring with a distributed force that will be denoted by $L_0$ pounds per circumferential inch. The magnitude of $L_0$ can be calculated in terms of $p$ from the standard "Equation [88]" based upon the theory of Von Sanden and Gunther [2,3] with appropriate correction for the faying flange if there is one. In the calculation as usually made, the subtended part of the shell is included along with the ring proper; it will be assumed that a suitable correction has been made so that $L_0$ represents only the radial force per inch acting on the toe of the web of the stiffener. Part of this force is transmitted through the web to the flange; let this part be denoted by $(1-\alpha) L_0$ pounds per inch. Thus the load carried by the web itself is $\alpha L_0$ pounds per inch. $L_0$ will be taken as a positive number whether the force is tensile or compressive.

Considerations of equilibrium show that the radial load $L_0$ evokes a total *compressive thrust* of $RL_0$ pounds on the whole cross section of the ring and that the part $(1-\alpha) R_f L_0$ of

this thrust is in the flange, $R$ being the radius of the toe of the web and $R_f$ the radius of the circular junction of web and flange. If the simplifying assumption is made that the compressive stress is uniform over the cross section of the ring, the ratio of the total ring thrust to the flange thrust is $(A_w + A_f)/A_f$, $A_w$ and $A_f$ being the cross-sectional areas of web and flange, respectively; and, if the difference between $R$ and $R_f$ is also neglected, this ratio equals $1/(1-\alpha)$. It follows that, approximately, for either inside or outside rings,

$$\alpha = \frac{A_w}{A_w + A_f} \qquad 1 - \alpha = \frac{A_f}{A_w + A_f} \tag{1a, b}$$

Here $A_w = ct_w$ in terms of the radial depth $c$ and the thickness $t_w$ of the web. More exact analysis given in Appendix A replaces $A_w$ and $A_f$ in Equations [1a, b] by $A'_w$ and $A'_f$, respectively, where, for inside rings,

$$A'_w = A_w - (1-\nu)\frac{c}{R}A_f \qquad A'_f = (1 + \frac{3}{2}\frac{c}{R})A_f \tag{1c, d}$$

where $\nu$ denotes Poisson's ratio. For outside rings, the terms containing $c/R$ have the opposite sign. For practical purposes, however, Equations [1a, b] should suffice.

The total thrust in the web itself is then $RL_0 - (1-\alpha)R_fL_0$ and division by $c$ gives $K_1$, the (mean) thrust per radial inch in the web. For an inside ring, $R_f = R - c$; for an outside ring, $R_f = R + c$ and $R$ should denote the outer radius of the cylinder. Hence, if also $K_f$ denotes the thrust in the flange,

$$K_1 = \alpha\frac{R}{c}L_0 + (1-\alpha)L_0 \qquad K_f = (1-\alpha)R_fL_0 \tag{2a, b}$$

except that for an outside ring the second term in $K_1$ becomes $-(1-\alpha)L_0$. This latter term may usually be omitted, however, since $R/c$ is a large number; the difference between $R$ and $R_f$ is also commonly ignored.

A certain complication arises from the fact that the local radial force in the web, denoted by $L$ pounds per circumferential inch, decreases progressively from $L_0$ at the shell to $(1-\alpha)L_0$ at the flange. Without serious error this variation may be assumed to be uniform over the depth of the web; then we can write for rings either inside or outside,

$$L = L_0(1 - \alpha\frac{z}{c}) \tag{3}$$

where $z$ denotes numerical distance from the shell (or the toe of the web).

The web will be treated as clamped at its junction with the shell. Actually, bending of the web will bend the cylinder as well, but it can be shown that the effect of this bending will be at least rather small provided the following inequalities hold:

$$\frac{c}{R} < \frac{1}{10} \qquad \frac{h}{t_w} > 3 \qquad \frac{R}{h} > 75$$

Here $h$ denotes the thickness of the shell and $t_w$ the thickness of the web; see Appendix B. A few formulas for the case of simple support of the web at the cylinder are given at the end of Appendix C.

The stress forces thus defined are related to tripping in the following ways:

1. The radial force $L$ in the web is tensile in an outside ring and so has a stabilizing influence, but in an inside ring this force tends to buckle the web in a radial direction.

2. The circumferential thrust $K_1$ in the web promotes buckling of the web (perpendicular to the radius) in circumferential waves. If the flange is either absent or extremely heavy, the critical thrust for such buckling of the web alone *decreases* as the number of waves increases until the shape of the waves becomes nearly square.

3. The thrust $K_f$ in the flange tends to buckle the flange in waves, the flange both bending in crosswise directions and rotating. The critical radial load for such buckling of the flange when unsupported by the web is least for buckling in two waves and is very small.

*Axisymmetric* buckling can result only from Action 1. In this type of buckling the flange plays a rather passive role, resisting turning of the radial tangent of the web at its junction with the flange. Buckling of the ring in *waves* on the other hand, results from a combination of the actions that lead in extreme cases to action of Type 2 or Type 3. Since, however, the web favors buckling in *many* waves whereas the flange prefers only *two*, the result for buckling of the ring as a whole is a compromise, with an intermediate number of waves and a critical load much higher than the minimum load for buckling of either web or flange alone.

There is also still another possibility, namely, that the flange alone might buckle by radial curling of its cross sections without deformation of its circumferential axis. It will be assumed that premature flange buckling of this sort is prevented by making the flange sufficiently thick.

Analysis of the flange itself requires only the theory of circular rings. With high fixity at the cylinder, however, bending of the web must necessarily occur, so that some degree of analysis of the web as a compressed plate cannot be avoided. The web will be considered first.

## DEFORMATION OF WEB

An accurate analysis of the web as a thin plate requires the use of polar coordinates and leads to very complicated formulas. In practical cases, however, the ratio $c/R$ is very small, and, for this reason, the *curvature* of the ring-shaped web has little effect beyond giving rise both to the circumferential thrust $K_1$ and to the variation with $z$ of the radial force $L$. As a plausible approximation, therefore, the web will be treated here as if it were a straight strip of width $c$ containing the perpendicular stress forces $L$ and $K_1$. It may be imagined, if desired, that radial equilibrium of the web element is preserved by the application of suitable fictitious distributed load forces, these replacing the net forces that arise in

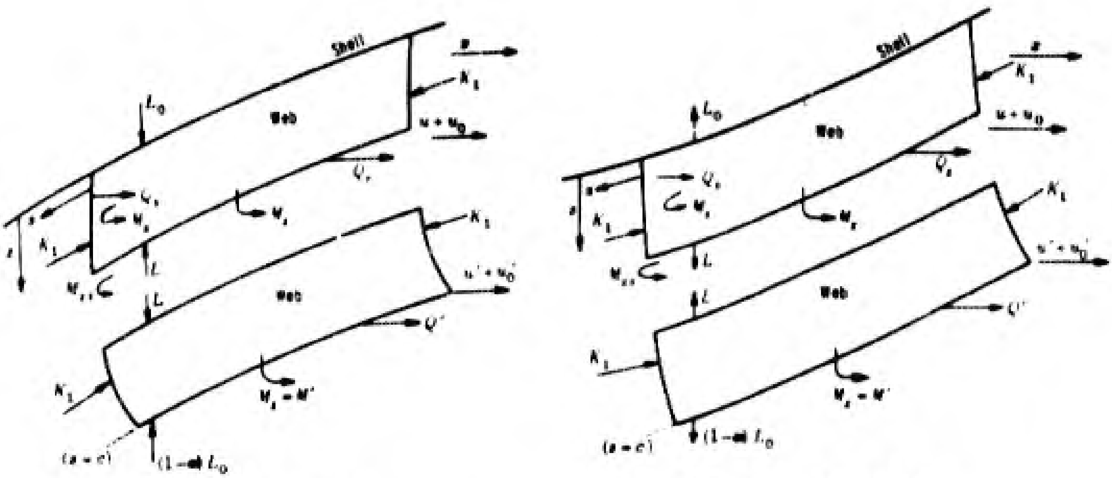

Figure 2a – Inside Ring

Figure 2b – Outside Ring

Figure 2 – Forces and Moments Acting in Web

reality from $K_1$ in combination with the curvature.

This assumption being made, the necessary analysis follows standard lines, as given, for example, in Sections 21 and 22 of Reference 4. The details are lengthy and will be relegated to Appendix C, only the skeleton of the argument being presented here.

Convenient coordinates are $z$, measured radially away from the shell, and a perpendicular rectangular coordinate $s$, actually equal to circumferential distance along the shell at the toe of the web; see Figure 2. Let the web have an initial small displacement $u_0 = u_0(z, s)$ and acquire an additional elastic displacement $u(z, s)$, both measured parallel to the cylinder axis from a transverse reference plane.

Consider a cross section of the web perpendicular to $z$. On the negative side of this cross section there act, per unit length of $s$, the radial stress force $L$, called positive in its actual direction under load, a shear force $Q_z$, perpendicular to the reference plane and positive in the same direction as $u$, a bending moment $M_z$, and a twisting moment $M_{zs}$. Both moments are taken to be positive when they tend to produce rotation from positive $z$ toward positive $u$. At the flange, however, or $z = c$, it is more convenient to replace $Q_z$ and $M_{zs}$ together in the usual way by a single shear force $Q'$ of magnitude

$$Q' = \left(Q_z - \frac{\partial M_{zs}}{\partial s}\right)_{z = c}$$

per unit length. The strains are thereby modified within a narrow border near $z = c$, but this local effect is unimportant.

Corresponding circumferential thrust, shear force, and moments, all per unit length, are denoted by $K_1$, $Q_s$, $M_s$, and $M_{zs}$.

In Appendix C (page 24) is indicated the derivation of the following differential equation for $u$ and the following expressions for $Q'$ and $M'$ or $M_z$ at $z = c$, for $M_0$ or $M_z$ at $z = 0$, and for $M_s'$ or $M_s$ at $z = c$, all for inside rings.

$$D\left(\frac{\partial^4 u}{\partial z^4} + 2\,\frac{\partial^4 u}{\partial s^2 \partial z^2} + \frac{\partial^4 u}{\partial s^4}\right) + \frac{\partial}{\partial z}\left[L\,\frac{\partial}{\partial z}(u + u_0)\right] + K_1\,\frac{\partial^2}{\partial s^2}(u + u_0) = 0 \qquad [4]$$

$$Q' = -\left\{D\left[\frac{\partial^3 u}{\partial z^3} + (2 - \nu)\,\frac{\partial^3 u}{\partial s^2 \partial z}\right] + L\,\frac{\partial}{\partial z}(u + u_0)\right\}_{z = c} \qquad [5]$$

$$M' = (M_z)_{z = c} = D\left(\frac{\partial^2 u}{\partial z^2} + \nu\,\frac{\partial^2 u}{\partial s^2}\right)_{z = c} \qquad [6a]$$

$$M_0 = (M_z)_{z = 0} = D\left(\frac{\partial^2 u}{\partial z^2}\right)_{z = 0} \qquad [6b]$$

$$M_s' = (M_s)_{z = c} = D\left(\frac{\partial^2 u}{\partial s^2} + \nu\,\frac{\partial^2 u}{\partial z^2}\right)_{z = c} \qquad [7]$$

$D$ is given by

$$D = \frac{E t_w^{\,3}}{12(1 - \nu^2)} \qquad [8]$$

where $E$ is Young's modulus, $\nu$ is Poisson's ratio, and $t_w$ is web thickness. It is assumed that $L$ and $K_1$ always act parallel to the reference plane in which the web lies initially; thus they are, in general, not quite perpendicular to the cross sections. In $Q'$ and $M'$, all derivatives are to be given their values at $z = c$.

Equation [4], being of the fourth order, requires four boundary conditions. Two conditions are provided at the shell or $z = 0$, where, according to the assumption previously stated, $u = 0$ and $\partial u/\partial z = 0$. The other two conditions are furnished indirectly by the connection to the flange at $z = c$.

It now turns out that, as is usual in such problems, if all quantities are expanded in Fourier series, the various terms satisfy all equations separately; it suffices, therefore, to study a single typical term. Proportionality to $\cos(ns/R)$ will be assumed, $n$ being an integer or zero. Then the equations become, for inside rings:

$$D\left(\frac{\partial^4 u}{\partial z^4} - 2\frac{n^2}{R^2}\frac{\partial^2 u}{\partial z^2} + \frac{n^4}{R^4}u\right) + \frac{\partial}{\partial z}\left[L\frac{\partial}{\partial z}(u + u_0)\right] - K_1\frac{n^2}{R^2}(u + u_0) = 0 \qquad [9]$$

$$Q' = -\left\{D\left[\frac{\partial^3 u}{\partial z^3} - (2-\nu)\frac{n^2}{R^2}\frac{\partial u}{\partial z}\right] + L\frac{\partial}{\partial z}(u + u_0)\right\}_{z = c} \qquad [10]$$

$$M' = (M_z)_{z = c} = D\left(\frac{\partial^2 u}{\partial z^2} - \nu\frac{n^2}{R^2}u\right)_{z = c} \qquad [11a]$$

$$M_0 = (M_z)_{z = 0} = D\left(\frac{\partial^2 u}{\partial z^2}\right)_{z = 0} \qquad [11b]$$

$$M_s' = (M_s)_{z = c} = D\left(-\frac{n^2}{R^2}u + \nu\frac{\partial^2 u}{\partial z^2}\right)_{z = c} \qquad [12]$$

For *outside* rings, $L$ is to be replaced by $-L$ in Equations [4] through [12], $L$ itself standing always for a positive number.

If $L$ did not vary with $z$ and if $u_0$ were of polynomial form in $z$, Equation [9] would be soluble in terms of trigonometric or hyperbolic functions. Even so, however, complicated formulas would be obtained. In any given case, a solution could easily be obtained by numerical integration. It was thought worthwhile, however, to look for approximate formulas that might be useful at least for survey purposes if not in practical application.

For this purpose, it is convenient to eliminate certain dimensions by writing, for *inside* rings:

$$y = \frac{z}{c} \qquad \beta = \frac{c}{R} > 0 \qquad [13a, b]$$

$$e = \frac{c^2 L_0}{D} - 2n^2\beta^2 \qquad \epsilon = \alpha\frac{c^2 L_0}{D} \qquad g = n^2\beta(\alpha\frac{c^2 L_0}{D} - n^2\beta^3) \qquad [14a, b, c]$$

For *outside* rings, it is convenient to keep $\beta$ positive; then $y$ is the same, as is also the formula for $g$, but by definition

$$e = -\frac{c^2 L_0}{D} - 2n^2\beta^2 \qquad \epsilon = -\alpha\frac{c^2 L_0}{D} \qquad [14d, e]$$

Furthermore, for practical use it will probably be sufficient to consider only the case of a simple tilt of the web at an angle $\theta$ which, like $u$, is proportional to cos $(ns/R)$. In this case

$$u_0 = \theta z = \theta_0 z \cos(ns/R) \qquad [15]$$

$\theta_0$ being a constant. (The case, $u_0 = (\theta_0 z + \gamma_0 z^2) \cos(ns/R)$ is only moderately more complicated.) Then

$$u_0 = \theta cy \qquad \frac{\partial u_0}{\partial y} = c\frac{\partial u_0}{\partial z} = \theta c \qquad \frac{\partial^2 u_0}{\partial y^2} = 0$$

With these substitutions and the introduction of $L = L_0(1 - \alpha y)$ from [3] and the approximate value $K_1 = \alpha R L_0/c$ from [2a], Equations [9] through [12] can be written as follows, for *inside* rings:

$$\frac{\partial^4 u}{\partial y^4} + e\frac{\partial^2 u}{\partial y^2} - gu - \epsilon\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) - \theta\alpha\frac{c^3 L_0}{D}(1 + n^2\beta y) = 0 \qquad [16]$$

$$Q' = -\frac{D}{c^3}\left[\frac{\partial^3 u}{\partial y^3} + (e - \epsilon + \nu n^2\beta^2)\frac{\partial u}{\partial y}\right]_{y=1} - (1-\alpha)\theta L_0 \qquad [17]$$

$$M' = \frac{D}{c^2}\left(\frac{\partial^2 u}{\partial y^2} - \nu n^2\beta^2 u\right)_{y=1} \qquad [18a]$$

$$M_0 = \frac{D}{c^2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \qquad [18b]$$

$$M_s' = \frac{D}{c^2}\left(-\frac{n^2}{R^2}u + \nu\frac{\partial^2 u}{\partial y^2}\right)_{y=1} \qquad [19]$$

A series solution of Equation [16] may now be sought. The series must begin with $y^2$ to make $u = \partial u/\partial y = 0$ at $y = 1$. The following series may be sufficiently extensive for practical purposes:

$$\begin{aligned} u = {} & A\left(y^2 - \frac{e}{12}y^4 + \frac{\epsilon}{30}y^5 + \frac{g+e^2}{360}y^6 - \frac{e\epsilon}{420}y^7\right) \\ & + B\left(y^3 - \frac{e}{20}y^5 + \frac{\epsilon}{40}y^6 + \frac{g+e^2}{840}y^7 - \frac{e\epsilon}{840}y^8\right) \qquad [20] \\ & + \frac{1}{24}\theta\alpha\frac{c^3 L_0}{D}\left[y^4 - \frac{e}{30}y^6 + \frac{2\epsilon}{105}y^7 + n^2\beta\left(\frac{1}{5}y^5 - \frac{e}{210}y^7 + \frac{\epsilon}{336}y^8\right)\right] \end{aligned}$$

Here the constants $A$ and $B$ represent the remaining two integration constants. By substitution it can be verified that the differential Equation [16] is satisfied through $Ay^3$, $By^4$, and $L_0 y^3$, and also that the terms shown would not be altered if the series were extended to higher powers of $y$.

The relation between web and flange is more easily represented, however, if $A$ and $B$ are replaced as unknown quantities by the values of u and $\partial u/\partial y$ at $y = 1$. This is easily done by differentiating Equation [20] three times and eliminating $A$ and $B$ from the four equations thus obtained; the result is two equations expressing $\partial^2 u/\partial y^2$ and $\partial^3 u/\partial y^3$ in terms of $U$ and $\partial u/\partial y$, all at $z = c$. Further explanation is given in Appendix C, pages 24–28.

The final formulas obtained by substitution for $\partial^2 u/\partial y^2$ and $\partial^3 u/\partial y^3$ in Equations [17], [18], and [19] may conveniently be written in the following form (for *inside* rings):

$$Q' = \frac{D}{c^3}\left(k_{11}\, u' - k_{12}\, c\omega\right) + (\alpha k_1 - 1)\,\theta L_0 \qquad [21]$$

$$M' = \frac{D}{c^2}\left(-k_{12}\, u' + k_{22}\, c\omega\right) + \alpha k_2 \theta c L_0 \qquad [22a]$$

$$M_0 = \frac{D}{c^2}\left(k_{31}\, u' - k_{32}\, c\omega\right) + \alpha k_3 \theta c L_0 \qquad [22b]$$

$$M'_s = \frac{D}{c^2}\left\{-\left[\nu k_{12} + (1-\nu^2)n^2\beta^2\right] u' + \nu k_{22} c\omega\right\} + \nu\alpha k_2 \theta c L_0 \qquad [23]$$

For *outside* ring ... terms have the opposite sign in these equations. Here $u'$ denotes the value of $u$ at $z$ ... whereas $\omega$ stands for the value of $\partial u/\partial z$ or $(1/c)\,\partial u/\partial y$ at $z = c$. The $k$'s are numerical ...nts whose values are

$$k_{11} = 12 - \frac{6}{5}e + \frac{3}{5}\epsilon - \frac{13}{35}g$$

$$k_{12} = 6 - \frac{e}{10} - \frac{11}{210}g + \nu n^2\beta^2$$

$$k_{22} = 4 - \frac{2}{15}e + \frac{\epsilon}{10} - \frac{g}{105} \qquad [24]$$

$$k_{31} = 6 - \frac{e}{10} + \frac{\epsilon}{10} + \frac{13}{420}g$$

$$k_{32} = 2 + \frac{e}{30} - \frac{\epsilon}{60} + \frac{g}{140}$$

$$k_1 = \frac{1}{2}\left(1 - \frac{\epsilon}{420} + 0.013\,g + 0.015\,e^2 - 0.025\,e\epsilon - \frac{7}{10}n^2\beta\right)$$

$$k_2 = \frac{1}{12}\left[1 + \frac{e}{60} - \frac{\epsilon}{105} - \frac{g}{140} - 0.008e^2 + 0.013\,e\epsilon \right.$$

$$\left. + \frac{3}{5}n^2\beta\left(1 + \frac{19e}{1260} + \frac{25\epsilon}{336}\right)\right]$$

$$k_3 = \frac{1}{12}\left[1 + \frac{e}{60} - \frac{\epsilon}{140} + \frac{2}{5}n^2\beta\,(1 + 0.019\,e - 0.009\,\epsilon)\right]$$

These formulas are valid for either inside or outside rings, but different formulas for $e$ and $\epsilon$ must be used, namely, [14a, b] for inside and [14d, e] for outside rings. (The absence of an $\epsilon$ term in $k_{12}$ is correct! Corresponding formulas for the case of simple support are given, in part, at the end of Appendix C.)

In practical cases, however, the quantities $e$, $\epsilon$, and $g$ are likely to be small enough so that many terms containing them can be omitted, especially from $k_1$, $k_2$, and $k_3$ or perhaps even from all $k$'s except $_{11}$. The principal reason for computing and retaining some of these terms in $k_1$, $k_2$, and $k_3$ was to show how small they are. If $e = \epsilon = g = 0$, the first six $k$'s take on the simple values that are obtained for a cantilever loaded at the end by a force $Q'$ and a moment $M'$:

$$k_{11} = 12 \qquad k_{12} = 6 \qquad k_{22} = 4 \qquad k_{31} = 6 \qquad k_{32} = 2$$

The factors $k_{11}D/c^3$, etc, that occur in [21], [22a, b], and [23] constitute stiffness coefficients for the force and moment that must be applied to the edge of the web at $s = c$ to produce assigned values of $u$ and $\omega$ at the edge, when the toe is fixed. More accurate values of these constants could be obtained by numerical integration. If, however, an electronic computer were to be used for this purpose, it would probably be preferable to correct the differential equation and other relations slightly by using polar coordinates. In any case, tabulation or plotting would be complicated by the fact that the $k$'s are functions of four dimensionless parameters, which may be listed as $c^2L_0/D$, $n$, $\beta$, and $a$.

Study of a few simple and easily integrable cases indicates that the expressions given here for $k_{11}$, $k_{12}$, and $k_{22}$ should be correct within at most a few percent for $|e| < 5$, $|\epsilon| < 5$, $|g| < 10$, and, of course, less in error for smaller values.

## NOTE ON OUTSIDE RINGS

When the ring is outside of the cylinder, it is convenient to draw $s$ away from the cylinder as before and therefore now outward instead of inward. Figures shown in this report may be adapted in thought to this case by assuming the cylinder axis to lie above the figure instead of below it and imagining all circumferential curvatures to be drawn upward instead

of downward; see Figure 2b. All curvature effects are reversed in sign. Furthermore, the radial force in the web is now tensile instead of compressive. A careful check shows that all web equations written in this report for inside rings remain true for outside rings provided $R$, $R_f$, $L$, $L_0$, and $\beta$ are replaced in them by $-R$, $-R_f$, $-L$, $-L_0$, and $-\beta$, respectively, the symbols themselves continuing to stand for positive numbers. Certain cases of this change have already been noted.

## DEFORMATION OF FLANGE

An *initial* deformation of the ring usually includes an initial deformation of the flange as well. Let the center of cross section of the flange have an initial small displacement $u_0'$ from the reference circle, $u_0'$ being a function of distance $s$ along the centroidal circle of the flange. (It will be noted that $s$ has slightly different meanings for the flange and web, but this difference will cause no confusion and is considered negligible.) The cross section may also be rotated slightly, but such an initial rotation, like small defects in shape, has little effect and may be neglected. Under load, the flange cross section may acquire both an additional elastic deflection $u'(s)$ parallel to the cylinder axis and an elastic rotation $\omega(s)$ about a circumferential axis, these deflections being equal to those denoted by the same symbols in the web theory; see Figure 3.

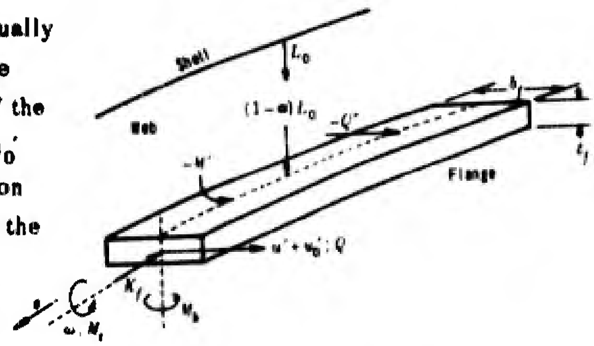


Figure 3 – Free-Body Diagram for a Section of Flange

The diagram is for an inside ring, for an outside ring, the curvature is upward and the direction of $L_0$ is reversed.

The flange is a circular ring loaded, per unit of its length, by the radial force $(1-\alpha)L_0$ and also by reactions to the other forces acting on the web, that is, by a force $-Q'$ positive in the same direction as $u'$ and a moment $M'$ positive when tending to increase $\omega$. (In Figure 3 positive directions for $-Q'$ and $-M'$ are indicated.)

On any flange cross section perpendicular to $s$ there act, besides the circumferential thrust $K_f$ evoked by the radial load, a shear force $Q$, taken positive in the same direction as $u'$, a bending moment $M_b$ tending when positive to increase the slope $du'/ds$, and a twisting moment $M_t$ tending when positive to increase $\omega$. Appropriate equations of equilibrium are developed on page 34 of Appendix D. It is found that the displacements $u' + u_0'$ and $\omega$ cause no tendency toward radial displacement, so that the tripping motion of the flange, like that of the web, is independent of deformations in a plane perpendicular to the axis of the cylinder.

For the case that all variables are proportional to cos $(ns/R_f)$, where $n$ is an integer and $R_f$ is the radius of the centroidal circle of the flange (or, nearly enough, the same as $R_f$

previously defined), it is found that (for inside rings)

$$-Q' = \frac{EI_f}{R_f^4} n^2(n^2u' + R_f\omega) + \frac{EJ_f}{R_f^4} n^2(u' + R_f\omega) - \frac{K_f}{R_f^2} n^2(u' + u_0') \qquad [25]$$

$$-M' = \frac{EI_f}{R_f^3} (n^2u' + R_f\omega) + \frac{EJ_f}{R_f^3} n^2(u' + R_f\omega) \qquad [26]$$

where $I_f$ is the areal moment of inertia of the flange cross section about a radial axis through its centroid and $J_f$ is the torsional rigidity of the flange divided by $E$. For a narrow rectangular cross section of width $b_f$ and thickness $t_f$

$$I_f = \frac{1}{12} b_f^3 t_f \qquad J_f = \frac{1}{6(1+\nu)} b_f t_f^3 \qquad [27a, b]$$

It will be noted that necessarily $Q' = 0$ when $n = 0$.

The formulas found in Appendix D for $M_b$ and for $(M_t)_{max}$ in the sinusoidal case may also be of use:

$$M_b = -\frac{EI_f}{R_f^2} (n^2u' + R_f\omega) \qquad [28a]$$

$$(M_t)_{max} = \frac{nEJ_f}{R_f^2} (u' + R_f\omega)_{max} \qquad [28b]$$

The unusual form of the latter equation is occasioned by the fact that $Q'$, $M'$, $M_b$, $u'$, and $\omega$ may all be assumed to vary in the same phase as $\cos(ns/R_f)$ but $M_t$ must then vary as $\sin(ns/R_f)$; however, Equation [28b] must hold for the maximum values as indicated.

As a check, it may be noted that Equations [25] through [28a, b] are all satisfied if

$$K_f = Q' = M' = M_b = M_t = 0 \qquad n = 1 \qquad u' = -R_f\omega$$

This is simply a free rigid rotation about an axis lying in the plane of the ring.

For *outside* rings, the only change required in Equations [25] through [28a, b] is to replace $R_f$ by $-R_f$ ($R_f^2$ being thus unaffected).

## DEFORMATION OF ENTIRE RING

Equations for the entire T-ring in the case considered in this report can now be obtained by setting $u_0' = u_0$ and equating the web expressions for $Q'$ and $M'$ to the flange expressions for the same quantities. For a simple initial tilt in $n$ waves as defined by Equation [15]

$$u_0 = \theta_c \qquad [29]$$

With this assumption, addition of [21] to [25] and of [22a] to [26] gives the two equations (for inside rings)

$$\left[\frac{n^2E}{R_f^4}(n^2I_f + J_f) + \frac{D}{c^3}k_{11} - n^2\frac{K_f}{R_f^2}\right]u + \left[n^2\frac{E(I_f + J_f)}{R_f^3} - \frac{D}{c^2}k_{12}\right]\omega$$

$$- n^2\frac{K_f}{R_f^2}\theta_c + (\alpha k_1 - 1)\theta L_0 = 0 \qquad [30a]$$

$$\left[\frac{E}{R_f^3}n^2(I_f + J_f) - \frac{D}{c^2}k_{12}\right]u' + \left[\frac{E}{R_f^2}(I_f + n^2J_f) + \frac{D}{c}k_{22}\right]\omega$$

$$+ \alpha k_2\theta c L_0 = 0 \qquad [30b]$$

Here, according to [2b], $K_f = (1 - \alpha)R_fL_0$.

A shortened notation, however, is more convenient in using these equations. Write

$$F = \frac{EI_f}{R_f^3} \qquad W = \frac{D}{c^2} \qquad \mu = \beta_f\frac{W}{F} \qquad \tau = \frac{J_f}{I_f} \qquad \beta_f = \frac{c}{R_f}$$

Here $F$, $W$, $\mu$, $\tau$, $\beta_f$ are all to be taken positive. The quantities $F$ and $W$ may be regarded as stiffness parameters of flange and web, respectively, each depending on the dimensions only of the relevant element; $\mu$ is their effective ratio. Similarly, $\tau$ is the ratio of twisting to bending stiffness of the flange. Then Equations [30a, b] can be written, after substituting for $K_f$ from [2b] and returning to $u'$ as in [29]:

$$\left[n^2(n^2 + \tau)\beta_fF + k_{11}W - n^2\beta_f(1 - \alpha)L_0\right]u' + \left[n^2(1 + \tau)F - k_{12}W\right]c\omega$$

$$= \left[n^2\beta_f(1 - \alpha) + 1 - \alpha k_1\right]\theta cL_0 \qquad [32a]$$

$$\left[n^2(1 + \tau)F - k_{12}W\right]u' + \left[(1 + n^2\tau)(F/\beta_f) + k_{22}W\right]c\omega = -\alpha k_2\theta cL_0 \qquad [32b]$$

For *outside* rings, $L_0$, $R_f$, $F$, and $\beta_f$ are to be replaced by $-L_0$, $-R_f$, $-F$, and $-\beta_f$, respectively, in Equations [30a, b] and [32a, b]. Also, in calculating the $k$'s from Formulas [24], Equations [14d, e] are to be used for $e$ and $\epsilon$.

Equations [32a, b] can be used to find the deformation of the ring when $L_0$ and an initial slope $\theta$ proportional to $\cos(ns/R)$ are given. Then the moments in the web, $M_0$ at the

toe and $M'$ and $M_s'$ at the flange, can be found by solving Equations [32a, b] for $u'$ and $c\omega$ and substituting these values in Equations [22a, b] and [23]. The bending moment in the flange can be found by substituting in Equation [28]. If $\theta c$ is assigned its maximum value (when cos $(ns/R) = 1$), the maximum values or amplitudes of the moments are obtained. Algebraic formulas could be written out, but for the case $n > 0$ it seems better to convert Equations [32a, b] to numerical form and then proceed numerically.

*Axisymmetric case.* If $n = 0$ and if $L_0$ is no larger than it usually is in practice, say $L_0 < W$, simple approximate formulas can be obtained. The contributions of $L_0$ to the $k$'s, which represents nonlinear effects, is then so small that it can be neglected except perhaps in the $e$ and $\epsilon$ terms in $k_{11}$. Then $k_{12} = 6$, $k_{22} = 4$, $k_{31} = 6$, $k_{32} = 2$, $k_1 = 1/2$, $k_2 = k_3 = 1/12$. Write

$$k_{11} = 12\left(1 - \frac{n}{4 + \frac{1}{\mu}}\right)$$

where

$$n = \pm\frac{1}{10}\left(4 + \frac{1}{\mu}\right)\left(e - \frac{1}{2}\epsilon\right) = \pm\frac{1}{10}\left(4 + \frac{1}{\mu}\right)\left(1 - \frac{\alpha}{2}\right)\frac{L_0}{W}$$

Then it is found by solving Equations [32a, b] that

$$u' = \frac{1}{6}\left[\frac{1 + 4\mu}{2(1 + \mu - n\mu)}\left(1 - \frac{2}{3}\alpha\right) + \frac{\alpha}{12}\right]\theta c\,\frac{L_0}{W}$$

$$c\omega = \frac{\mu}{2(1 + \mu - n\mu)}\left(1 - \frac{2}{3}\alpha\right)\theta c\,\frac{L_0}{W}$$

after dropping a small term $\alpha n/[6(1 + 4\mu)]$ that occurs added to $1 - 2\alpha/3$ in both $u'$ and $c\omega$; and, approximately,

$$M' = -\frac{1}{2(1 + \mu - n\mu)}\left(1 - \frac{2}{3}\alpha\right)\theta c L_0 \qquad [33]$$

$$M_0 = \left[\frac{1 + 2\mu}{2(1 + \mu - n\mu)}\left(1 - \frac{2}{3}\alpha\right) + \frac{\alpha}{6}\right]\theta c L_0 \qquad [34]$$

Here $\mu$ is commonly not far from unity; $\mu = 0$ corresponds to a rigid flange, and $\mu = \infty$ corresponds to no flange at all.

*Maximum stresses* in web and flange can be estimated in the usual way from the moments. The compressive membrane stress in the web in the *circumferential* direction is $\sigma_c = K_1/t_w = \alpha R L_0/(ct_w)$, to sufficient accuracy. In comparison with this, circumferential bending stresses in the web due to wave formation by bending may be neglected. The

corresponding component of radial stress in the web $\sigma_r$ decreases from $L_0/t_w$ at the shell to $(1-\alpha)L_0/t_w$ at the flange, being compressive in an inside ring but tensile in an outside ring. The maximum radial bending stress $\sigma_b$ will presumably occur at one edge of the web and can be estimated as

$$\sigma_b = \frac{6}{t_w^{\,2}}\left|M_{max}\right| \qquad [35]$$

where $|M_{max}|$ is the larger of $|M_1|$ or $|M_0|$ as given by [22a, b].

The most critical side of the web is that on which the radial bending stress is tensile and thus opposite in sign to the compressive stress $\sigma_c$. On this side of the web of an *inside* ring, the principal stresses, defined with tension positive, are

$$\sigma_1 = \sigma_b - \sigma_r \qquad \sigma_2 = -\sigma_c \qquad \sigma_3 = 0$$

Here $\sigma_b$, $\sigma_r$, and $\sigma_c$ all have positive values. Substitution in the Hencky-Von Mises yield criterion of $(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2\sigma_y^{\,2}$, $\sigma_y$ being the ordinary yield stress, then gives as the criterion of incipient yielding in an inside ring

$$(\sigma_b-\sigma_r)^2+\sigma_c(\sigma_b-\sigma_r)+\sigma_c^2 = \sigma_y^2 \qquad [36a]$$

or, if $\sigma_r$ is dropped because of its relatively small magnitude,

$$\sigma_b^2+\sigma_c^2+\sigma_b\sigma_c = \sigma_y^2 \qquad [36b]$$

If $\sigma_b = \sigma_c$, it is necessary only that $\sigma_b = 0.58\,\sigma_y$ for yielding to occur.

In an *outside* ring, $-\sigma_r$ is replaced by $+\sigma_r$ in these equations; there is no other change.

The maximum stress in the flange in simply

$$(1-\alpha)\frac{R_f L_0}{A_f}+\frac{6}{b_f^{\,2}}\left|M_b\right| \qquad [37]$$

where $b_f$ is the (axial) width of the flange, $A_f$ its cross-sectional area, and $M_b$ can be calculated from [28].

## LATERAL BUCKLING OF PERFECT RING

The critical load for lateral buckling of an ideal T-ring can be estimated as the value of $L_0$ that is obtained by setting $\theta = 0$ in Equations [32a, b], whereupon the equations become homogeneous in $u'$ and $c\omega$, and then equating the determinant of the coefficients to zero. The general relationships are best exhibited if the resulting equation for $L_0$ is written as follows:

$$(k_{11}k_{22} - k_{12}^2)W^2 + \left\{\frac{k_{11}}{\beta_f} + n^2\left[\frac{\tau k_{11}}{\beta_f} \pm 2(1+\tau)k_{12} + \beta_f(n^2+\tau)k_{22}\right]\right\}FW$$

$$+ n^2(n^2-1)^2\tau F^2 - n^2\left[\beta_f k_{22}W + (1+n^2\tau)F\right](1-\alpha)L_0 = 0 \quad [38]$$

Where $\pm$ occurs in this equation, the upper sign is to be taken for inside rings and the lower sign for outside rings, other terms being the same for both except for differences in the values of the $k$'s. All letters stand for positive quantities (except possibly $k_{11}$, $k_{12}$, or $k_{22}$).

The $W^2$ term in Equation [38] represents direct resistance to buckling of the web itself; and the $F^2$ term, direct resistance of the flange. The $FW$ term represents additional buckling resistance due to incompatibility between the preference of the web for many waves and the preference of the flange for very few waves. The $L_0$ term, finally, represents the contribution to buckling by the flange thrust $K_f$. The corresponding effect of the web thrust $K_1$ is included in the $k$'s.

Since the load $L_0$ occurs implicitly in the $k$'s as well as explicitly in the last term of Equation [38], this equation can be solved for $L_0$ only by successive approximation. Fortunately this procedure is facilitated by the facts that $k_{11}$ varies much more rapidly with $L_0$ than do $k_{12}$ and $k_{22}$, and the $k$'s become relatively less important when $n$ is large. For purposes of computation, it appears to be convenient to write Equation [38] in the form (dividing through by $n^2WF$):

$$(1-\alpha)(1+\mu k_{22}+n^2\tau)\frac{L_0}{W} = \left[(k_{11}k_{22}-k_{12}^2)\mu + k_{11}\right]\frac{1}{n^2\beta_f} + \frac{\tau}{\beta_f}k_{11}$$

$$\pm 2(1+\tau)k_{12} + (n^2+\tau)\beta_f k_{22} + (n^2-1)^2\frac{\tau}{\mu}\beta_f \quad [39]$$

Here $\mu = \beta_f W/F$ and is always positive.

Three special cases deserve comment.

1. *Axisymmetric Buckling.* For $n = 0$, Equation [38] reduces to

$$(k_{11}k_{22} - k_{12}^2)\mu + k_{11} = 0 \quad [40a]$$

Thus in this case buckling results solely from variation of the $k$'s with increasing load.

Insertion of Formulas [24] for the $k$'s and use of [14a, b] for $e$ and $\epsilon$ give the following equation, which is easily solved for $L_0$, representing the critical radial load for axisymmetric buckling of an inner ring:

$$\left[5.2 - 3.6\alpha + \frac{1}{\mu}(1.2 - 0.6\alpha)\right]\frac{c^2 L_0}{D} = 12\left(1 + \frac{1}{\mu}\right) \qquad [40b]$$

$$+ (0.15 - 0.2\alpha + 0.06\alpha^2)\left(\frac{c^2 L_0}{D}\right)^2$$

Values of $L_0$ obtained from this formula agree well with those obtained from the formulas given in Reference 1.

2. *Web with No Flange.* If the flange is ommitted, then $\alpha = 1$ and $F = 0$ and Equation [38] becomes simply

$$k_{11} k_{22} - k_{12}^2 = 0 \qquad [41]$$

The web is now comparable with the clamped-free plate under longitudinal compression that is treated on page 341 of Reference 5. If in the differential Equation [16] $\theta$ is set equal to 0 (for the buckling case) and $\alpha$ equal to 1, and if the $L_0$ terms in the formulas for $e$ and $\epsilon$ are dropped (to be justified later), the equation becomes

$$\frac{\partial^4 u}{\partial y^4} - 2n^2\beta^2 \frac{\partial^2 u}{\partial y^2} + \left(n^4\beta^4 - n^2\beta \frac{c^2 L_0}{D}\right) u = 0$$

This is of the same form as Equation [c] on page 338 of Reference 5. Since, however, the free boundary in this case occurs at $y = 1$, $b$ must be set equal to 1 in Reference 5. Then the solution for buckling under minimum load, as inferred from Figure 178 and Equation [j] of Reference 5, becomes, in our notation,

$$n^2\beta^2 = 3.69 \qquad n^2\beta \frac{c^2 L_0}{D} = \text{``}\frac{N_x}{D}\,\frac{m^2\pi^2}{a^2}\text{''} = 48.4$$

On the other hand, if Equation [41] is solved for $L_0$ using

$$e = -2n^2\beta^2 \qquad g = n^2\beta(c^2 L_0/D) - n^4\beta^4$$

and then $n^2\beta^2$ is chosen so as to make $L_0$ a minimum, the result is

$$n^2\beta^2 = 3.90 \qquad n^2\beta \frac{c^2 L_0}{D} = 61.6$$

The rough agreement between these values of $n^2\beta^2$ and of the $L_0$ term with the exact values as inferred from Reference 5 may be regarded as encouraging, since here $e = -7.8$, $g = 61.6 - 15.2 = 46.4$, whereas $|e| < 5$ and $|g| < 10$ have been proposed as reasonable limits

for the use of our approximate formulas for the $k$'s.

The values of $n^2\beta^2$ and of $n^2\beta c^2 L_0/D$ as inferred from Reference 5 give $(c^2 L_0/D) = 13.1\,\beta$. Since usually $\beta < 0.1$, this makes the term that was dropped in $e$ less than 1.3 and so fairly small relative to $-n^2\beta^2$ or $-7.8$. The web thus approximates in buckling fairly well to a straight strip under longitudinal compression (as is intuitively obvious). A rough estimate of the minimum buckling load $L_0$ on the web without flange and of the associated value of $n$ may thus be made from the equations

$$L_0 = 13.1\,\beta \frac{D}{c^2} \qquad n = \frac{1.92}{\beta} \tag{42a, b}$$

3. *Flange Alone (under Uniform Radial Load).* When the web is absent, so that $\alpha = 0$, then $W = 0$ and Equation [38] gives, for $n > 1$,

$$L_{on} = \frac{(n^2 - 1)^2 \tau}{1 + n^2 \tau} \frac{EI_f}{R_f^3} \tag{43}$$

$L_{on}$ represents here the radial load per unit length that is required for lateral buckling (without radial displacement) on the assumption that the applied load remains exactly parallel to the initial plane during buckling. $L_{on}$ increases rapidly with increasing $n$.

The minimum value of $L_{on}$, for $n = 2$, is relatively low. For the stiffener flange without web on an SS 212-Class submarine $L_{on}$ is only 5.7 lb/in. By contrast, as estimated from Equation [42a], something like 3000 lb/in. of radial load would be required to buckle the web without the flange. Thus, in this case, the flange badly needs the web, but the web could dispense with the flange entirely. (For the SS 212 stiffener, $R = 96$ in., $R_f = 101$ in., $c = 5.0$ in., $t_w = 0.343$ in., $\beta = 0.052$, $D = 1.11 \times 10^5$, $b_f = 3.44$ in., $t_f = 3/8$ in., $I_f = 1.273$, and $\tau = 0.0182$.)

## AN EXAMPLE

To illustrate the use of the formulas, calculations were made for the inside ring that is described on page 17 of Reference 1. The value of $L_0$, the load per inch on the toe of the web, is represented there by $\delta F$. The relevant parameters for this case are, in the present notation and in terms of pounds and inches,

$\alpha = 0.414$
$R = 41.7$ in.
$\beta = c/R = 0.0525$
$D = 5500$
$\tau = 0.0125$*
$W = D/c^2 = 1149$
$\mu = 0.927$

$c = 2.19$ in.
$R_f = 39.4$ in.
$\beta_f = c/R_f = 0.0556$
$I_f = 0.1405$
$J_f = 0.001757$
$F = 68.9$

*Inadvertently, the value $\tau = 0.0124$ was used in calculating.

(a) *Bending Moments Due to an Initial Load.* In Reference 1, calculations were made for radial load $L_0$ on the ring of 363 lb/in. and with the web tilted uniformly by 3 deg. The same conditions were assumed for the present calculations, the maximum tilt being 3 deg when $n>0$. Thus, for maximum conditions, $\theta = 0.0524$ radians and $\theta c L_0 = 41.6$. For $n = 0$, Equations [33] and [34] where used; for $n>0$, Equations [32a, b] were simplified by dropping $\tau$ where it is simply added to 1 or $n^2$ and then solving these equations for $u'$ and $c\omega$ and substituting in [22a, b] or [23] to find the moments $M'$ (denoted by $M_d$ in Reference 1), $M_0$, and perhaps $M_s'$. The highly variable constant $k_{11}$ was calculated in detail; for the other $k$'s the following formulas were used:

$$k_{12} = 6 + 0.00083\,n^2;\ k_{22} = 4$$

$$k_{31} = 6;\ k_{32} = 2;\ k_1 = 0.5 - 0.0184 n^2$$

$$k_2 = \frac{1}{12}(1 + 0.0315 n^2);\ k_3 = \frac{1}{12}(1 + 0.021 n^2)$$

The calculated values obtained for the moments in the web were as follows, certain values obtained in Reference 1 from the "second approximation" being added in parentheses:

| $n$ | 0 | 10 | 16 | 24 |
|---|---|---|---|---|
| $M'$ | −8.3(−9.07) | −21.4 | −15.7 | 3.8 |
| $M_0$ | 25.7(25.6) | 29.7 | 30.2 | 23.8 |
| $M_s'$ | | −7.1 | 6.1 | |

The nonlinear effect on $M_0$, when $n = 0$, is about +6 percent.

These results invite the tentative conclusion that, for engineering purposes, it will probably suffice to calculate moments in the web by the simple formulas that hold when $n = 0$, using either Equations [33] and [34] of the present report or the formulas in Reference 1. The associated stresses are discussed in Reference 1 for the case $n = 0$. Their maximum values will differ so little for the other values of $n$ that it was not thought worthwhile to compute them.

It may be of interest to note that, if the flange is removed, $M_0$ decreases from 26.7 to 21.7 when $n = 0$ but increases from 30 to 223 when $n = 16$. Thus, regarded as a means of minimizing the magnitude of $M_0$ due to initial tilt, the flange is worse than useless when $n = 0$ but is very helpful, as would be expected, when $n$ is so large that the circumferential thrust becomes important.

(b) *Buckling.* In a perfect ring of the usual proportions, failure by yield occurs long before there is any semblance of an approach to lateral elastic buckling. Nevertheless the theoretical interest of the buckling phenomenon in a T-ring was considered sufficient to justify a few calculations, in spite of their more tedious nature. The appropriate buckling

equation was solved by trial, Equations [40a] for $n = 0$, and Equation [39] for $n > 0$. To illustrate the difference between an inside and an outside ring, calculations were made also for a ring of the same cross-sectional characteristics placed outside of a cylinder of the same radius ($R_f$ then changing from 39.4 in. to 44 in.).

The buckling loads thus found were as follows:

| | | | | | |
|---|---|---|---|---|---|
| *Inside ring:* | $n = 0$ | 2 | 5 | 7 | 10 |
| | $\frac{L_0}{W} = 5.8$ | 5.6 | 5.8 | 6.3 | 8.9 |
| *Outside ring:* | $n = 6$ | 8 | 10 | 12 | |
| | $\frac{L_0}{W} = 10.3$ | 6.4 | 6.3 | 8.6 | |

Here $W = 1.15 \times 10^3$ lb/in. Evidently the inside ring prefers to buckle with $n = 2$ or 3, the outside one with $n$ equal to about 9, but the minimum buckling loads are not much different. For the inside ring, the minimum buckling load is about the same as that calculated for $n = 0$, a case that is more readily treated by the method of Reference 1.

## APPENDIX A

## STRESSES IN A PERFECT T-RING

Equations [1c, d] can be deduced from certain equations in Reference 6. For the radial stress $\sigma_r$ and the radial displacement $w$ (inward) in the web, Equation [44] on page 59 and Equation [53] on page 67 of Reference 6 give, after substituting $B = H = K = 0$ and $u = -w$,

$$\sigma_r = \frac{A}{r^2} + 2C \qquad Ew = -(1+\nu)\frac{A}{r} + 2(1-\nu)Cr$$

Denote by $\sigma_{ro}$, $\sigma_{rf}$ the values of $\sigma_r$ at $r = R$ and $r = R_f$, respectively. At $r = R_f$, $w$ is also the inward displacement of the flange and evokes in it a circumferential compressive strain $w/R_f$ and an associated thrust $K_f = EA_f w/R_f$. Thus we have the three equations

$$\sigma_{ro} = \frac{A}{R^2} + 2C \qquad \sigma_{rf} = \frac{A}{R_f^2} + 2C$$

$$\frac{R_f K_f}{A_f} = Ew = -(1+\nu)\frac{A}{R_f} + 2(1-\nu)CR_f$$

From these three equations, $A$ and $C$ can be eliminated. Introduce also, for an inside ring, $R_f = R - c$, and, from [2b] $K_f = (1-\alpha)R_f L_0 = (1-\alpha)R_f t_w \sigma_{ro}$, $t_w$ being the web thickness, and, from the definition of $\alpha$, $t_w \sigma_{rf} = (1-\alpha)L_0 = (1-\alpha)t_w \sigma_{ro}$, also $ct_w = A_w$, the cross-sectional area of the web. Then it is found that

$$\alpha = \frac{(R-c)(2R-c)A_w - (1-\nu)(2R-c)cA_f}{(R-c)(2R-c)A_w + [2R^2 - (1-\nu)(2R-c)c]A_f}$$

After dividing out the coefficient of $A_w$, expanding in powers of $c/R$ in numerator and denominator separately, and keeping only the first power of $c/R$, it is found that approximately

$$\alpha = \frac{A_w - (1-\nu)\left(\frac{c}{R}\right)A_f}{A_w + \left[1 + \left(\frac{1}{2} - \nu\right)\left(\frac{c}{R}\right)\right]A_f}$$

This can be written in the form expressed by Equations [1c, d] in the present report. For an *outside* ring, it is only necessary to change the sign of $c$ throughout.

## APPENDIX B

## RELATIVE STIFFNESS OF SHELL

In tripping, a stiffening ring will usually exert a moment on the shell tending to bend it around a transverse circumference. To judge whether such yielding of the shell makes it unreasonable to assume, as an approximation, fixity of the web at the shell, as has been done in this report, the stiffness of the shell against such bending should be compared with the stiffness of the ring for similar bending. Since, however, this comparison would require a considerable investigation, similar comparisons will be made here, first with the stiffness of the web alone for bending relative to the flange by distributed moments applied to its toe and then with the stiffness of the flange for similar bending by moments applied along its length.

The axisymmetric case of bending of a long cylinder by a moment $M_0$ per unit length applied along an interior transverse circumference can be inferred from Equations [236] on page 393 of Reference 4. Here $x$ denotes longitudinal distance along the cylinder. Put $x = 0$, and also $w = 0$ in the first of these equations, and then eliminate $Q_0$ between the first two equations; also substitute $M_0/2$ for "$M_0$" in the equations, since our cylinder is equivalent to two half cylinders connected together. The stiffness of the cylindrical shell $S_{sh}$ defined as moment per unit of circumferential length over slope of generator produced is thus found to be, when $n = 0$,

$$S_{sh} = M_0/(dw/dx)_{x=0} = \text{"}4\beta D\text{"} = \frac{Eh^3}{[3(1-\nu^2)]^{3/4}} \frac{1}{\sqrt{hR}} \qquad [44]$$

The general case $n > 0$ is rather complicated. For our purpose it may be adequate to note that, for large $n$, the cylinder will bend nearly like a flat plate, for which the differential equation for normal deflection $w$ may be written

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial s^2} + \frac{\partial^4 w}{\partial s^4} = 0$$

or, for variation of $w$ as $\cos(ns/R)$,

$$\frac{\partial^4 w}{\partial x^4} - 2\frac{n^2}{R^2}\frac{\partial^2 w}{\partial x^2} + \frac{n^4}{R^4} w = 0 \qquad [45]$$

A solution of this equation that may represent bending by distributed couples $M_0 \cos(ns/R)$ applied along a line at $x = 0$, is, if $C$ is written temporarily for $\cos(ns/R)$,

for $x > 0$:

$$\frac{w}{C} = Axe^{-nx/R}$$

$$\frac{1}{C}\frac{\partial w}{\partial x} = Ae^{-nx/R}\left(1 - \frac{nx}{R}\right)$$

$$\frac{1}{C}\frac{\partial^2 w}{\partial x^2} = Ae^{-nx/R}\left(-2\frac{n}{R} + \frac{n^2 x}{R^2}\right)$$

for $x < 0$:

$$\frac{w}{C} = Axe^{nx/R}$$

$$\frac{1}{C}\frac{\partial w}{\partial x} = Ae^{nx/R}\left(1 + \frac{nx}{R}\right)$$

$$\frac{1}{C}\frac{\partial^2 w}{\partial x^2} = Ae^{nx/R}\left(2\frac{n}{R} + \frac{n^2 x}{R^2}\right)$$

Since the bending moment per unit of $s$-length is

$$M = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial s^2}\right) = \frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} - \nu\frac{n^2}{R^2}w\right)$$

there is a discontinuity in $M$ at $x = 0$ of the magnitude

$$4\frac{nA}{R}\frac{Eh^3}{12(1-\nu^2)}C$$

This must equal $M_0 C$. Also $AC = (\partial w/\partial x)_0$. Hence when $n > 0$, approximately

$$S_{sh} = M_0 C/(\partial w/\partial x)_0 = \frac{Eh^3}{3(1-\nu^2)}\frac{n}{R} \qquad [46]$$

The actual value of $S_{sh}$ will be greater than that calculated from either [44] or [46].

For a minimum estimate of $S_{sh}$, therefore, the smaller of the two values of $S_{sh}$ given by [44] and [46] may be used in any given case.

For comparison between the shell and the web of the stiffener, it may be noted that the stiffness of the web, treated as if it were a straight strip, when it is clamped at the flange and bent by couples $M_0$ per unit length distributed uniformly around the toe, is found by elementary reasoning to be (when $n = 0$)

$$S_w = M_0/(dw/dx)_0 = \frac{D}{c} = \frac{Et_w^{\,3}}{12(1-\nu^2)c} \qquad [47]$$

$c$ being the radial depth of the web and $t_w$ its thickness. If, on the other hand, the applied couples vary as $\cos(ns/R)$, $S_w$ increases with $n$ but slowly at first; if the waves become of length $2c(n = \pi R/2c)$, $S_w$ is increased in the ratio 1.87. This number was obtained by studying the following solution of Equation [45] with $n/R$ replaced by $q$:

$$u = C_1\, qx \sinh(qx) + C_2(qx \cosh qx - \sinh qx)$$

To obtain a corresponding formula for the flange, we may assume $u = 0$ in the flange formulas, as was assumed for the shell. Then Equation [26] gives for the rotational stiffness of the flange

$$S_f = (-M_1)/w = \frac{E}{R_f^{\,2}}(I_f + n^2 J_f) \qquad [48]$$

Comparison of these formulas leads, after slight simplification, to the conclusion that

$$\frac{S_{sh}}{S_w} > B_1 \frac{c}{R}\left(\frac{h}{t_w}\right)^3 \qquad \frac{S_{sh}}{S_f} > B_2 \frac{R_f h^3}{I_f + n^2 J_f} \qquad [49a, b]$$

where $B_1$ is the greater of $5\sqrt{R/h}$ or $2n$ and $B_2$ the greater of $0.5\sqrt{R/h}$ or $n/3$. In general, the ratio $S_{sh}/S_w$ should be satisfactorily large to justify the assumption of fixity at the shell. However, if $S_{sh}/S_f > S_{sh}/S_w$, a smaller value of $S_{sh}/S_w$ as given by [49a] is acceptable, since then the flexibility of the ring as a whole is considerably reduced by flexibility of the flange.

## APPENDIX C

## APPROXIMATE ANALYSIS OF WEB AS STRAIGHT PLATE

Further details of the treatment of the web will be given here. The reader will be assumed to have read the discussion of the web preceding Equation [4]; the notation is illustrated again, for an inside ring, in Figure 4a.

The usual formulas for the three moments previously defined are

$$M_z = D\left(\frac{\partial^2 u}{\partial z^2} + \nu \frac{\partial^2 u}{\partial s^2}\right) \qquad M_s = D\left(\frac{\partial^2 u}{\partial s^2} + \nu \frac{\partial^2 u}{\partial z^2}\right)$$

$$M_{zs} = M_{sz} = D(1-\nu)\frac{\partial^2 u}{\partial s \partial z} \qquad D = \frac{Et_w^{\,3}}{12(1-\nu^2)}$$

Here $t_w$ is the thickness of the web.

Since the shear forces $Q_z$ and $Q_s$ are assumed always to be perpendicualr to the reference plane and $L$ and $K_1$ always parallel to it, radial translational equilibrium of the elements requires no consideration. Translational equilibrium of a $dz\,ds$ element of the web in the $u$-direction, and its rotational equilibrium about axes parallel to $s$ and $z$, respectively, require that

$$ds\left(dz\frac{\partial Q_z}{\partial z}\right) + dz\left(ds\frac{\partial Q_s}{\partial s}\right) = 0$$

$$ds\left(dz\frac{\partial M_z}{\partial z}\right) + dz\left(ds\frac{\partial M_{sz}}{\partial s}\right) + dz(Q_z ds) + (L ds)dz\frac{\partial}{\partial z}(u+u_0) = 0$$

$$dz\left(ds\frac{\partial M_s}{\partial s}\right) + ds\left(dz\frac{\partial M_{zs}}{\partial z}\right) + ds(Q_s dz) + (K_1 dz)ds\frac{\partial}{\partial s}(u+u_0) = 0$$

Here the $(Lds)$ term arises from the fact that the opposing $L$ and $L + dL$ forces on the $ds$ sides of the element act along lines spaced $dz\,\frac{\partial}{\partial z}(u+u_0)$ apart and so exert a turning moment; see Figure 4b. The $K_1$ term arises in a similar way. Hence, dividing by $dz\,ds$, we have as the three equations of equilibrium for an element of the web:

$$\frac{\partial Q_z}{\partial z} + \frac{\partial Q_s}{\partial s} = 0$$

$$\frac{\partial M_z}{\partial z} + \frac{\partial M_{sz}}{\partial s} + Q_z + L\frac{\partial}{\partial z}(u+u_0) = 0$$

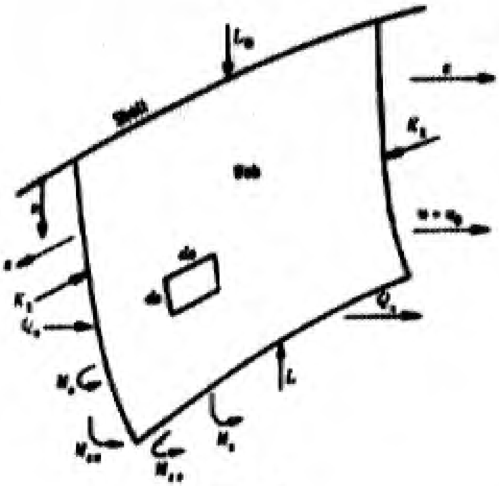

Figure 4a

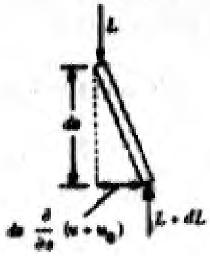

Figure 4b

**Figure 4 – Forces and Moments Acting in Web**

$$\frac{\partial M_s}{\partial s} + \frac{\partial M_{zs}}{\partial z} + Q_s + K_1 \frac{\partial}{\partial s}(u + u_0) = 0$$

From these three equations, a single equation free of $Q_z$ and $Q_s$ is easily obtained; and substitution in this equation of the expressions previously quoted for the $M$'s then gives the differential equation written in the text as Equation [4].

Substitution for $Q_z$ and $M_{sz}$ in the defining equation for $Q'$ also gives Equation [5]; and the expressions for $M_z$ at $z = c$ and at $z = 0$ (where $u = 0$) were copied as Equations [6] and [7].

(The main text should now be read from Equation [4] through the paragraph containing Equation [20].)

The series written in Equation [20] was obtained by substituting a series for $u$ in Equation [16] and balancing coefficients of each power of $y$ in the usual way. It turns out that the coefficients of $y^2$ and $y^3$ remain arbitrary, so that the series can be written in the form shown in Equation [20]. The coefficients of the power of $y$ as far as they are shown are easily seen to be uniquely determined.

By differentiating this series three times and then setting $y = 1$, the following expressions are obtained for $u$ and its derivatives at $y = 1$:

$$u = A\left(1 - \frac{e}{12} + \frac{\epsilon}{30} + \frac{g + e^2}{360} - \frac{e\epsilon}{420}\right) + B\left(1 - \frac{e}{20} + \frac{\epsilon}{40} + \frac{g + e^2 - e\epsilon}{840}\right)$$

$$+ \frac{1}{24} L'\left[1 - \frac{e}{30} + \frac{2\epsilon}{105} + \frac{1}{5} n^2 \beta \left(1 - \frac{e}{42} + \frac{\epsilon}{67.2}\right)\right]$$

$$\frac{\partial u}{\partial y} = A\left(2 - \frac{e}{3} + \frac{\epsilon}{6} + \frac{g + e^2 - e\epsilon}{60}\right) + B\left(3 - \frac{e}{4} + \frac{3\epsilon}{20} + \frac{g + e^2}{120} - \frac{e\epsilon}{105}\right)$$

$$+ \frac{1}{6} L'\left[1 - \frac{e}{20} + \frac{\epsilon}{30} + \frac{1}{4} n^2 \beta \left(1 - \frac{e}{30} + \frac{\epsilon}{42}\right)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = A\left(2 - e + \frac{2}{3}\epsilon + \frac{g + e^2}{12} - \frac{e\epsilon}{10}\right) + B\left(6 - e + \frac{3}{4}\epsilon + \frac{g + e^2}{20} - \frac{e\epsilon}{50}\right)$$

$$+ \frac{1}{2} L'\left[1 - \frac{e}{12} + \frac{\epsilon}{15} + \frac{1}{3} n^2 \beta \left(1 - \frac{e}{20} + \frac{\epsilon}{24}\right)\right]$$

$$\frac{\partial^3 u}{\partial y^3} = A\left(-2e + 2\epsilon + \frac{g + e^2}{3} - \frac{e\epsilon}{2}\right) + B\left(6 - 3e + 3\epsilon + \frac{g + e^2}{4} - \frac{2}{5} e\epsilon\right)$$

$$+ L'\left[1 - \frac{e}{6} + \frac{\epsilon}{6} + \frac{1}{2} n^2 \beta \left(1 - \frac{e}{12} + \frac{\epsilon}{12}\right)\right]$$

where $L' = \theta\alpha \dfrac{c^3 L_0}{D}$. It is also useful to note that

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = c^2 \left(\frac{\partial^2 u}{\partial z^2}\right)_{z=0} = 2A$$

$A$ and $B$ were then eliminated from the four equations containing them, and all fractions thereby introduced were expanded in powers and products of $e$, $\epsilon$, and $g$. Only $e$, $\epsilon$, $g$, $e^2$, and $e\epsilon$ were retained, however, since higher powers or products would be furnished also by the omitted higher powers of $y$ in the series for $u$, whose coefficients remain unknown. Furthermore, in the final expressions for $\partial^2 u/\partial y^2$ and $\partial^3 u/\partial y^3$, certain terms in $e^2$, $e\epsilon$, or $g$ had coefficients less than 1/400 as large as the leading numeric to which they were added, or, in terms containing $L_0$ less than 1/100 as large, and most of such terms were dropped.

The result was the following approximate equations, in which a subscript 1 distinguishes values at $y = 1$ and a subscript 0 values at $y = 0$:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_1 = -\left(6 - \frac{e}{10} - \frac{11}{210} g\right)(u)_1 + \left(4 - \frac{2}{15} e + \frac{\epsilon}{10} - \frac{g}{105}\right)\left(\frac{\partial u}{\partial y}\right)_1$$

$$+ \frac{1}{12} \theta\alpha \frac{c^3 L_0}{D}\left[1 + \frac{e}{60} - \frac{\epsilon}{105} - \frac{g}{140} - 0.005 e^2\right.$$

$$\left. + 0.013 e\epsilon + \frac{3}{5} n^2 \beta \left(1 + \frac{19e}{1260} + \frac{25\epsilon}{336} - 0.014 e^2\right)\right]$$

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_1 = -\left(12 - \frac{6}{5}e + \frac{3}{5}\epsilon - \frac{13}{35}g\right)(u)_1 + \left(6 - \frac{11}{10}e + \epsilon - \frac{11}{210}g\right)\left(\frac{\partial u}{\partial y}\right)_1$$

$$+ \frac{1}{2}\theta\alpha\frac{c^3 L_0}{D}\left[1 + \frac{\epsilon}{420} - \frac{17}{1260}g - 0.015e^2 + 0.025e\epsilon + \frac{7}{10}n^2\beta\right]$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = 2A = \left(6 - \frac{e}{10} + \frac{\epsilon}{10} + \frac{13}{420}g\right)(u)_1 - \left(2 + \frac{e}{30} - \frac{\epsilon}{60} + \frac{g}{140}\right)\left(\frac{\partial u}{\partial y}\right)_1$$

$$+ \frac{1}{12}\theta\alpha\frac{c^3 L_0}{D}\left[1 + \frac{e}{60} - \frac{\epsilon}{140} + \frac{2}{5}n^2\beta(1 + 0.019e - 0.009\epsilon)\right]$$

Substitution of the first two of these equations for $\partial^2 u/\partial y^2$ and $\partial^3 u/\partial y^3$ in Equations [17] and [18a] gives Equations [21] and [22a] for the relations at $y = 1$. (In these equations, $u'$ and $\omega$ effectively replace $A$ and $B$ as the remaining unknown constants.) Finally, substitution of the series found for $(\partial^2 u/\partial y^2)_0$ in Equation [18b] gives Equation [22b] and Equation [19] becomes Equation [23]. (Reading of the text below Equation [23] may now be resumed.)

## FORMULAS FOR SIMPLE SUPPORT AT $z = 0$

If simple support is assumed to occur where the web joins the shell, so that

$$u = \frac{\partial^2 u}{\partial z^2} = 0 \text{ at } z = 0$$

then the series for $u$ is found to be, through $y^7$:

$$u = A\left(y + \frac{\epsilon}{24}y^4 + \frac{g}{120}y^5\right) + B\left(y^3 - \frac{e}{20}y^5 + \frac{\epsilon}{40}y^6 + \frac{g}{840}y^7\right)$$

$$+ \theta\alpha\frac{c^3 L_0}{D}\left[\frac{1}{24}y^4 - \frac{e}{720}y^6 + \frac{\epsilon}{1260}y^7 + n^2\beta\left(\frac{y^5}{120} - \frac{ey^7}{5040}\right)\right]$$

In this case the series was terminated after terms linear in $e$, $\epsilon$, and $g$. It was then found that the constants in Equations [21] and [22] are as follows:

$$k_{11} = 3 - \frac{6}{5}e + \frac{3}{8}\epsilon - \frac{17}{35}g$$

$$k_{12} = 3 - \frac{1}{5}e - \frac{3}{35}g + \nu n^2\beta^2$$

$$k_{22} = 3 - \frac{1}{5}e + \frac{1}{8}\epsilon - \frac{2}{105}g$$

$$k_1 = \frac{3}{8}\left[1 - \frac{e}{90} - \frac{\epsilon}{280} - \frac{16}{15}\, n^2\beta\left(1 - \frac{59}{672}\, e\right)\right]$$

$$k_2 = \frac{1}{8}\left[1 + \frac{e}{30} - \frac{\epsilon}{70} + \frac{8}{15}\, n^2\beta\left(1 - \frac{e}{14}\right)\right]$$

Here, however, errors may rise to 5 or 10 percent if either $|e|$ or $|\epsilon|$ exceeds 2, or if $|g| > 10$.

## APPENDIX D

## ANALYSIS OF FLANGE

The analysis required for the flange is a slight modification of that given for a circular strip in Section 34 of Reference 5. Positive directions for the quantities defined ahead of Equation [25] in the present report are shown again at the top of Figure 5. As usual, an inside ring will be under consideration except as stated.

The bending moment $M_b$ in the flange arises in part from its curvature in a direction perpendicular to its plane, of magnitude $d^2u'/ds^2$, $u_0'$ not contributing. There is also, however, a contribution from a variable rotation $\omega$. This rotation occurs about a circumferential axis but, at cross sections $A$ and $B$ separated by a distance $ds$, the circumferential axes are inclined to each other at the angle $ds/R_f$; see Figure 5. Since small rotations can be treated as vectors, the rotation $\omega + d\omega$ at $B$ can be resolved into a major component about an axis parallel to the axis drawn for $\omega$ at $A$, and a minor component about a perpendicular axis ($aa'$ in Figure 5) of magnitude $(\omega + d\omega)\, ds/R_f$. The major component, in combination with the rotation at $A$, does not bend the ring. The perpendicular component, however, not being matched by a similar

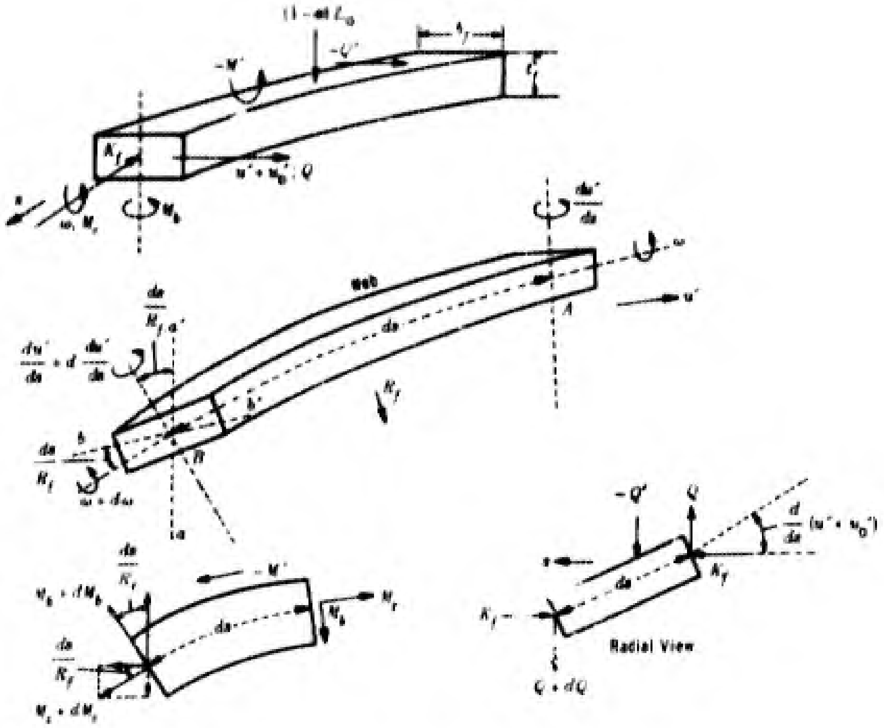


Figure 5 – Force Actions or Displacements for Section of Flange

These diagrams are drawn for an inside ring.

rotation at $A$, gives to the ring a component of curvature in the $u'$ direction of magnitude $-(\omega + d\omega)/R_f$. Thus, dropping $d\omega$,

$$M_b = EI_f\left(\frac{d^2u'}{ds^2} - \frac{\omega}{R_f}\right) \qquad [50]$$

where $I_f$ denotes the areal moment of inertia of the flange cross section about a radial axis (perpendicular to the cylinder axis).

Similarly, the cross section at $A$ undergoes a rotation $du'/ds$ about a radial axis through its centroid, but the corresponding rotation at $B$ has, besides a major component about an axis parallel to the radius at $A$ which can produce only bending, a minor component about an axis $bb'$ parallel to the circumferential axis at $A$. This component, being unbalanced by a similar rotation at $A$, twists the flange. Thus the total rate of twist is

$$\frac{d\omega}{ds} + \frac{1}{R_f}\frac{du'}{ds}$$

Hence

$$M_t = EJ_f\left(\frac{d\omega}{ds} + \frac{1}{R_f}\frac{du'}{ds}\right) \qquad [51]$$

where $J_f$ is the torsional rigidity of the flange (or torsional moment divided by the rate of twist) divided by $E$.

The laws of static equilibrium may now be brought forward. Translational equilibrium perpendicular to the initial plane of the ring requires that

$$\frac{dQ}{ds} - Q' = 0 \qquad [52]$$

There is a turning moment on the $ds$ section about a radial axis of magnitude $dM_b$. Another turning component arises from the twisting moment $M_t$. Couples, like rotations, can be treated as vectors; hence they are represented in a right-handed manner by arrows in the lower part of Figure 5. It is clear that the difference in direction of $M_t$ at the two ends of the section results in a net turning moment of magnitude (to the first order)

$$-M_t\frac{ds}{R_f}$$

Another turning moment about the radius is $Q\,ds$ due to $Q$, and still another arises from the circumferential thrust $K_f$ of magnitude

$$K_f\,ds\left[\frac{d}{ds}(u' + u_0')\right]$$

see Figure 5 at bottom. Equating the sum of all these radial turning moments to zero and dividing by $ds$ gives

$$\frac{d}{ds} M_b - \frac{1}{R_f} M_t + Q + K_f \frac{d}{ds} (u' + u_0') = 0 \qquad [53]$$

In a similar way the bending moment gives rise to a twisting component $M_b \, ds/R_f$; adding to this both $dM_t$ and the external twisting moment $-M' ds$, equating the sum to 0, and dividing by $ds$:

$$\frac{d}{ds} M_t + \frac{1}{R_f} M_b - M' = 0 \qquad [54]$$

These equations can now be made to yield relations between $Q'$ and $M'$ on the one hand and $u'$, $u_0'$, $\omega$ on the other. To eliminate $Q$, differentiate Equation [53] with respect to $s$ and then substitute $Q' = -dQ/ds$ according to [52]; then substitute from Equations [50] and [51] for $M_b$ and $M_t$. The results are the useful formulas:

$$-Q' = EI_f \left( \frac{d^4 u'}{ds^4} - \frac{1}{R_f} \frac{d^2 \omega}{ds^2} \right) - \frac{EJ_f}{R_f} \left( \frac{d^2 \omega}{ds^2} + \frac{1}{R_f} \frac{d^2 u'}{ds^2} \right) + K_f \frac{d^2}{ds^2} (u' + u_0') \qquad [55]$$

$$-M' = -\frac{EI_f}{R_f} \left( \frac{d^2 u'}{ds^2} - \frac{\omega}{R_f} \right) - EJ_f \left( \frac{d^2 \omega}{ds^2} + \frac{1}{R_f} \frac{d^2 u'}{ds^2} \right) \qquad [56]$$

For the sinusoidal case in which

$$\frac{d^2}{ds^2} \to -\frac{n^2}{R^2}$$

these equations become Equations [25] and [26] in the text. Equation [50] also becomes Equation [28a]. The equations require that $Q'$, $M'$, and $M_b$ be in phase with $u'$ and $\omega$ around the ring; if, however, these quantities are all proportional to $\cos ns/R$, Equation [51] makes $M_t$ proportional to $\sin ns/R$. All the equations of this report hold whether the sinusoidal symbols including derivatives are assumed to represent values at some given $s$ or maximum values occurring around the flange. For practical use, it is convenient to replace Equation [51] in the sinusoidal case by the following relation between maximum values cited in the text as Equation [28b]:

$$(M_t)_{max} = \frac{neJ_f}{R_f^2} (u' + R_f \omega)_{max}$$

For an *outside* ring, retention of the assumption that the web lies above the flange in

all figures requires that the curvature be shown as upward instead of downward. The angle $d\alpha/R_f$ is thereby replaced by $-d\alpha/R_f$; no other change is required. Hence the equations that have been obtained for the flange all remain valid provided $R_f$ is replaced throughout by $-R_f$ ($R_f^2$ being thus unchanged).

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